

Proving the First Sen Conjecture

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Abstract

Great progress has been made over the last decade in finding analytical solutions to open string field theory. These notes offer a quick, but detailed introduction to open string field theory and discuss analytical methods in the context of proving the first Sen conjecture.

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1 Open String Field Theory

This section is intended to provide an introduction to the mathematics of open string field theory. It should be accessible to a reader comfortable with string theory and introductory conformal field theory at the level of the first three chapters in Polchinski [1].

1.1 String Fields

Given a string background with spacetime fields ϕ^i , one can find a corresponding 2-Dimensional Conformal Field Theory (CFT). We will focus on the bosonic open string, for which the CFT can be represented as a product of a $c = 26$ CFT in the matter sector and a $c = -26$ CFT in the ghost sector [2].

$$CFT = (CFT)_{\text{matter}} \otimes (CFT)_{\text{ghost}} \quad (1)$$

The ghosts are 2-Dimensional ghosts arising from reparametrization invariance of the world sheet. This CFT has an infinite dimensional vector space of states, or Hilbert Space, denoted by \mathcal{H} . We say Φ is a *string field* if

$$\Phi = \sum_i \Phi_i \phi^i, \quad \Phi_i \in \mathcal{H}. \quad (2)$$

1.2 BRST Operator

Consider the energy-momentum tensor in the matter sector

$$T^m(z) \equiv - : \partial X^\mu \partial X_\mu : (z). \quad (3)$$

Given the BRST current

$$j_B \equiv cT^m(z) + : bc\partial c : + \frac{3}{2}\partial^2 c(z), \quad (4)$$

we create the BRST operator Q_B [3], defined as

$$Q_B \equiv \oint \frac{dz}{2\pi i} j_B(z). \quad (5)$$

From this point on, we use the cleaner notation

$$Q_B \equiv Q. \quad (6)$$

Consider string fields Φ , Ψ , and the operator Q . One can show

$$Q^2 = 0 \quad (7)$$

$$Q(\Phi + \Psi) = Q\Phi + Q\Psi. \quad (8)$$

1.3 Grassmann Number

We need to become familiar with the concept of a *Grassmann Number* [4]¹. However, a formal definition is not important for our purposes. We will instead focus on properties of the Grassmann number. An explicit way to denote Grassmann number is to define $\epsilon(\Phi)$ to be the Grassmann number of a string field Φ . In practice, such notation is almost never used. The standard convention is that whenever Φ appears as an exponent, it is understood that we mean $\epsilon(\Phi)$. For example,

$$(-1)^\Phi \equiv (-1)^{\epsilon(\Phi)} . \quad (9)$$

Given the BRST operator Q , one can show

$$\epsilon(Q\Phi) = \epsilon(\Phi) + 1 . \quad (10)$$

The Grassmann number of a product of string fields Φ, Ψ is given by

$$\epsilon(\Phi \star \Psi) = \epsilon(\Phi) + \epsilon(\Psi) . \quad (11)$$

From now on, we use $\epsilon(\Phi)$ only when there may be an ambiguity.

1.4 Correlators

In order to get a map of the type $\mathcal{H} \otimes \cdots \rightarrow \mathbb{R}$, we will need to use a *correlator* [5]. Consider some elements $\Phi_i, \Phi_j \in \mathcal{H}$. If Φ_i has dimension h_i and is appropriately normalized, then the correlator of Φ_i and Φ_j in the upper half-plane is given by

$$\langle \Phi_i(x) \Phi_j(y) \rangle_{UHP} = \frac{\delta_{ij}}{(x-y)^{2h_i}} . \quad (12)$$

If we have three elements of \mathcal{H} , then the correlator is given by

$$\langle \Phi_i(x) \Phi_j(y) \Phi_k(z) \rangle_{UHP} = \frac{C_{ijk}}{(x-y)^{h_i+h_j-h_k} (x-z)^{h_i+h_k-h_j} (y-z)^{h_j+h_k-h_i}} , \quad (13)$$

where C_{ijk} are the OPE coefficients. Next explore two maps of the type $\mathcal{H} \otimes \cdots \rightarrow \mathbb{R}$ that will be useful in many aspects of open string field theory.

¹It should be noted that in [4], the convention is used that the Grassmann number of a string field is even. Later, we use the convention that the Grassmann number of a string field is odd, in order to best agree with modern literature. Either choice is acceptable, as long as one is consistent.

1.5 BPZ Inner Product

Given $\Phi_i(z), \Phi_j(z) \in \mathcal{H}$, the BPZ inner product $\langle \Phi_i, \Phi_j \rangle$ is [6] a map $\mathcal{H} \otimes \mathcal{H} \rightarrow \mathbb{R}$, defined as

$$\langle \Phi_i, \Phi_j \rangle \equiv \langle f \circ \Phi_i(0) \Phi_j(0) \rangle_{UHP} , \quad (14)$$

where f is the conformal map

$$f(\xi) = -\frac{1}{\xi} . \quad (15)$$

Here are some properties of the BPZ inner product that will be useful for our purposes. Consider some elements of \mathcal{H} and the BRST operator Q . It can be shown [4] that

$$\langle \Phi_i, \Phi_j \rangle = (-1)^{\Phi_i \Phi_j} \langle \Phi_j, \Phi_i \rangle \quad (16)$$

$$\langle Q\Phi_i, \Phi_j \rangle = -(-1)^{\Phi_i} \langle \Phi_i, Q\Phi_j \rangle \quad (17)$$

From these rules, we can prove some nice identities. First,

$$\begin{aligned} \langle \Phi_i, Q\Phi_j \rangle &= (-1)^{\Phi_i(\Phi_j+1)} \langle Q\Phi_i, \Phi_j \rangle \\ &= \text{sign}[(-1)^{\Phi_i(\Phi_j+1)}] \langle Q\Phi_i, \Phi_j \rangle , \end{aligned} \quad (18)$$

so Q is self adjoint up to a sign. Next, consider a string field Φ and note that $\epsilon[\Phi](\epsilon[\Phi] + 1)$ is always an even number. We have

$$\begin{aligned} \langle \Phi, Q\Phi \rangle &= (-1)^{\Phi(\Phi+1)} \langle Q\Phi, \Phi \rangle \\ &= \langle Q\Phi, \Phi \rangle . \end{aligned} \quad (19)$$

According to (17), we also have

$$\langle Q\Phi, \Phi \rangle = -(-1)^\Phi \langle \Phi, Q\Phi \rangle , \quad (20)$$

and it follows that $\epsilon(\Phi)$ is an odd number for all string fields. We say that Φ is Grassmann odd.

1.6 Star Product

It is sufficient for our purposes to define the star product, denoted by \star , through its action inside the BPZ inner product. Given $\Phi_i(z), \Phi_j(z), \Phi_k(z) \in \mathcal{H}$, we can create a map $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \rightarrow \mathbb{R}$, often called the *three-vertex*, given by

$$\langle \Phi_i, \Phi_j \star \Phi_k \rangle = \langle f_0 \circ \Phi_i(0) f_1 \circ \Phi_j(0) f_2 \circ \Phi_k(0) \rangle_{UHP} , \quad (21)$$

where f_1, f_2 , and f_3 are conformal maps. For our purposes it will be convenient [5] to use the conformal maps

$$f_0(\xi) = \tan \left[\frac{2}{3} \left(\arctan \xi + \frac{\pi}{2} \right) \right] \quad (22)$$

$$f_1(\xi) = \tan\left(\frac{2}{3}\arctan\xi\right) \quad (23)$$

$$f_2(\xi) = \tan\left[\frac{2}{3}\left(\arctan\xi - \frac{\pi}{2}\right)\right]. \quad (24)$$

We can express the above functions more compactly as

$$f_n(\xi) = \tan\left[\frac{2}{3}\left(\arctan\xi - \frac{(n-1)\pi}{2}\right)\right]. \quad (25)$$

Two useful identities that can be shown from the above definitions are

$$\langle\Phi_i, \Phi_j \star \Phi_k\rangle = \langle\Phi_i \star \Phi_j, \Phi_k\rangle \quad (26)$$

$$Q(\Phi_i \star \Phi_j) = (Q\Phi_i) \star \Phi_j + (-1)^{\Phi_i}\Phi_i \star (Q\Phi_j). \quad (27)$$

Be careful when manipulating star products, noting that the star product is *not* in general commutative

$$\Phi \star \Psi \neq \Psi \star \Phi. \quad (28)$$

1.7 Witten's Action

Here we explore the origin of the open string field theory equation of motion for a string field Ψ , given by

$$Q\Psi + \Psi \star \Psi = 0. \quad (29)$$

We use units of $\hbar = c = \alpha' = 1$, and continue to do so for the remainder of these notes. Witten's open string field theory action [7] is

$$S = -\frac{1}{g_o^2} \left[\frac{1}{2} \langle\Psi, Q\Psi\rangle + \frac{1}{3} \langle\Psi, \Psi \star \Psi\rangle \right], \quad (30)$$

where g_o is the open string coupling constant, the inner product is the BPZ inner product, Q is the BRST operator, \star is the star product. Let us derive the classical equation of motion (29) from $\Psi \rightarrow \Psi + \delta\Psi$.

$$S + \delta S = -\frac{1}{g_o^2} \left[\frac{1}{2} \langle\Psi + \delta\Psi, Q(\Psi + \delta\Psi)\rangle + \frac{1}{3} \langle\Psi + \delta\Psi, (\Psi + \delta\Psi) \star (\Psi + \delta\Psi)\rangle \right] \quad (31)$$

Any term or inner product with two or more factors of $\delta\Psi$ will be zero. Since we want $\delta S = 0$, we can also get rid of the factor of $-\frac{1}{g_o^2}$ and obtain

$$\delta S = \frac{1}{2} \langle\Psi, Q\delta\Psi\rangle + \frac{1}{2} \langle\delta\Psi, Q\Psi\rangle + \frac{1}{3} \langle\Psi, \Psi \star \delta\Psi\rangle + \frac{1}{3} \langle\Psi, \delta\Psi \star \Psi\rangle + \frac{1}{3} \langle\delta\Psi, \Psi \star \Psi\rangle \quad (32)$$

It is in this form we finally get an intuition for the strange factors of $\frac{1}{2}$ and $\frac{1}{3}$ appearing in the action. They are put in because we want the pre-factors of the Q inner products and the \star inner products to independently add up to one. Continue simplifying, noting that Ψ is Grassmann odd. It turns out that the Grassmann number of $\delta\Psi$ is irrelevant. Obtain the expression

$$\begin{aligned} 0 &= \langle Q\Psi, \delta\Psi \rangle + \langle \Psi \star \Psi, \delta\Psi \rangle \\ &= \langle Q\Psi + \Psi \star \Psi, \delta\Psi \rangle . \end{aligned} \tag{33}$$

Since this must be true for any $\delta\Psi$, it follows that

$$Q\Psi + \Psi \star \Psi = 0 , \tag{34}$$

and we have the desired classical equation of motion. It can similarly be shown that this equation has gauge invariance $\delta\Psi = Q\Lambda + \Psi \star \Lambda - \Lambda \star \Psi$.

2 The First Sen Conjecture

2.1 D-brane Tension

Here we derive the tension \mathcal{T}_{25} of a D-25-brane [8]. First consider two parallel D-p-branes separated by a distance Y^μ . These two D-p-branes can interact by exchanging closed strings. We can then calculate the coupling T_p of the closed strings to the D-branes and divide a factor of the string-string coupling g_s to obtain our desired \mathcal{T}_p . The coupling T_p can be calculated with help from the one-loop vacuum amplitude given by

$$\ln(Z_{\text{vac}}) = -\frac{V_D}{2} \int \frac{d^D k}{(2\pi)^D} \ln(k^2 + M^2) , \tag{35}$$

which is often referred to as the *Coleman-Weinberg formula*. However, since

$$-\frac{1}{2} \ln(k^2 + M^2) = \int_0^\infty \frac{dt}{2t} e^{-\frac{(k^2 + M^2)t}{2}} , \tag{36}$$

we can write

$$Z_{\text{vac}} \equiv \mathcal{A} = V_D \int \frac{d^D k}{(2\pi)^D} \int_0^\infty \frac{dt}{2t} e^{-\frac{(k^2 + M^2)t}{2}} . \tag{37}$$

If the mass spectrum is given by

$$M^2 = \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i - 1 + \frac{Y \cdot Y}{4\pi^2} , \tag{38}$$

then the one-loop vacuum amplitudes may be thought of as a sum of the zero point energies of all the modes

$$\mathcal{A} = V_{p+1} \int \frac{d^{p+1} k}{(2\pi)^{p+1}} \int_0^\infty \frac{dt}{2t} \sum_I e^{-2\pi t(k^2 + M_I^2)} , \tag{39}$$

where the sum over I covers the entire physical spectrum. If we define

$$q = e^{-2\pi t}$$

$$f(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n), \quad (40)$$

then we can compute the sum over the spectrum and obtain

$$\mathcal{A} = 2V_{p+1} \int_0^{\infty} \frac{dt}{2t} (8\pi^2 t)^{-\frac{p+1}{2}} e^{-\frac{Y \cdot Y t}{2\pi}} f(q)^{-24}. \quad (41)$$

$$\begin{aligned} \mathcal{A} &\sim V_{p+1} \frac{24}{2^{12}} (4\pi^2)^{11-p} \pi^{\frac{p-23}{2}} \Gamma\left[\frac{23-p}{2}\right] |Y|^{p-23} \\ &= V_{p+1} \frac{24\pi}{2^{10}} (4\pi^2)^{11-p} G_{25-p}(Y^2), \end{aligned} \quad (42)$$

where $G_d(Y^2)$ is the masses scalar Green's function in d dimensions. Next we calculate \mathcal{A} from a field theory calculation of the exchange of a graviton and a dilation between the two D-branes. In this case, we obtain

$$\mathcal{A} \sim \frac{D-2}{4} V_{p+1} T_p^2 \kappa_o^2 G_{25-p}(Y^2). \quad (43)$$

Therefore, we find that

$$T_p = \frac{\sqrt{\pi}}{16\kappa_o} (4\pi^2)^{\frac{11-p}{2}}. \quad (44)$$

If we divide by a factor of the string coupling g_s and note that $\kappa = \kappa_o g_s$, the D-p-brane Tension is given by

$$\mathcal{T}_p = \frac{\sqrt{\pi}}{16\kappa} (4\pi^2)^{\frac{11-p}{2}}. \quad (45)$$

We also have

$$\frac{2\pi g_o^2}{\kappa} = 2^{18} \pi^{\frac{25}{2}}, \quad (46)$$

so it follows that

$$\mathcal{T}_{25} = \frac{1}{2\pi^2 g_o^2}. \quad (47)$$

2.2 Claim

Claim of Sen's First Conjecture: The energy density of the true vacuum found by solving the open string field theory equations of motion should be equal to minus the tension of the D25 brane [5, 9, 10, 11].

Our task is to prove

$$E = -\mathcal{T}_{25} = -\frac{1}{2\pi^2 g_o^2}. \quad (48)$$

3 Solution in the Schnabl Gauge

3.1 The Schnabl Gauge

Consider the following operators

$$L \equiv L_0 = \oint \frac{dz}{2\pi i} z T(z) \quad (49)$$

$$B = \oint \frac{dz}{2\pi i} z b(z) \quad (50)$$

$$c(z) = \sum_{n=-\infty}^{\infty} \frac{c_n}{z^{n-1}}, \quad (51)$$

where c_n is an L eigenstate with eigenvalue n . Unless otherwise specified, the path taken in a closed loop integral is around the unit circle in the complex plane. We can considerably simplify the star product [5, 12] by making the conformal transformation

$$z \rightarrow f(z) = \arctan(z). \quad (52)$$

Now, we have

$$\mathcal{L} = \oint \frac{dz}{2\pi i} (1+z^2) \arctan(z) T(z) = L_0 + \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{4k^2-1} L_{2k} \quad (53)$$

$$\mathcal{B} = \oint \frac{dz}{2\pi i} (1+z^2) \arctan(z) b(z) = b_0 + \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{4k^2-1} b_{2k} \quad (54)$$

$$\tilde{c}(\tilde{z}) = \sum_{n=-\infty}^{\infty} \frac{\tilde{c}_n}{\tilde{z}^{n-1}}, \quad (55)$$

where $\tilde{z} = \arctan(z)$ and \tilde{c}_n is an \mathcal{L} eigenstate with eigenvalue n . One can show

$$\{Q, b_n\} = L_n \quad (56)$$

$$[Q, L_n] = 0. \quad (57)$$

It follows that

$$\{Q, \mathcal{B}\} = \mathcal{L}. \quad (58)$$

Historically, it was quite difficult to find an analytical solution to (29). However, many papers have been written discussing numerical solutions. One of the major developments of open string field theory was the discovery and clever use of the *Schnabl Gauge*

$$B\Psi = 0. \quad (59)$$

We use (58) and (59) to put the equation of motion in a more convenient form.

$$\begin{aligned}
0 &= Q\Psi + \Psi \star \Psi \\
&= \mathcal{B}Q\Psi + \mathcal{B}(\Psi \star \Psi) \\
&= (Q\mathcal{B} + \mathcal{B}Q)\Psi + B(\Psi \star \Psi) \\
&= \mathcal{L}\Psi + \mathcal{B}(\Psi \star \Psi)
\end{aligned} \tag{60}$$

Next we explore the first analytical solution found using this gauge.

3.2 Schnabl's Analytical Solution

Begin with the equation of motion written in the form

$$\mathcal{L}\Psi + \mathcal{B}(\Psi \star \Psi) = 0. \tag{61}$$

Start solving this equation level by level, and a clear pattern develops. It looks as if the solution Ψ is given by

$$\begin{aligned}
\Psi &= \sum_{n=0}^{\infty} \sum_{\substack{p=-1 \\ p \text{ odd}}}^{\infty} \frac{\pi^p}{2^{n+2p+1}n!} (-1)^n B_{n+p+1} (\mathcal{L} + \mathcal{L}^\dagger)^n \tilde{c}_{-p} |0\rangle \\
&+ \sum_{n=0}^{\infty} \sum_{\substack{p,q=-1 \\ p+q \text{ odd}}}^{\infty} \frac{\pi^{p+q}}{2^{n+2(p+q)+3}n!} (-1)^{n+q} B_{n+p+q+2} (\mathcal{B} + \mathcal{B}^\dagger) (\mathcal{L} + \mathcal{L}^\dagger)^n \tilde{c}_{-p} \tilde{c}_{-q} |0\rangle,
\end{aligned} \tag{62}$$

where B_n are the Bernoulli numbers and $|0\rangle$ is the $SL(2, \mathbb{R})$ vacuum. If

$$\psi_n = \frac{2}{\pi^2} U_{n+2}^\dagger U_{n+2} \left[\left(\mathcal{B} + \mathcal{B}^\dagger \right) \tilde{c} \left(-\frac{\pi}{4} n \right) \tilde{c} \left(\frac{\pi}{4} n \right) + \frac{\pi}{2} \left(\tilde{c} \left(-\frac{\pi}{4} n \right) + \tilde{c} \left(\frac{\pi}{4} n \right) \right) \right] |0\rangle \tag{63}$$

$$U_n^\dagger U_n = \sum_{m=0}^{\infty} \frac{1}{m!} \left(\frac{2-n}{2} \right)^m (\mathcal{L} + \mathcal{L}^\dagger)^m, \tag{64}$$

then it is possible to put Ψ in a more manageable form

$$\Psi = \lim_{N \rightarrow \infty} \left[\psi_N - \sum_{n=0}^N \partial_n \psi_n \right]. \tag{65}$$

It is shown [5] that Ψ satisfies (29) and appropriately describes the true vacuum.

3.3 Proof of the First Conjecture

We normalize the correlators such that $V_{25} = 1$, so no distinction is made between energy and energy density. In that case, the energy density of a static configuration is minus the action, so we want to show

$$E = -S[\Psi] = \frac{1}{g_0^2} \left[\frac{1}{2} \langle \Psi, Q\Psi \rangle + \frac{1}{3} \langle \Psi, \Psi \star \Psi \rangle \right]. \quad (66)$$

Since Ψ satisfies the equation of motion, it follows that

$$E = \frac{1}{6g_0^2} \langle \Psi, Q\Psi \rangle. \quad (67)$$

Therefore, we only need to prove the simplified statement

$$\langle \Psi, Q\Psi \rangle = -\frac{3}{\pi^2}. \quad (68)$$

By making use of the quantities

$$\begin{aligned} \mathcal{C}_0 &= \lim_{N \rightarrow \infty} \langle \psi_N, Q\psi_N \rangle = \frac{1}{2} + \frac{2}{\pi^2} \\ \mathcal{C}_1 &= \lim_{N \rightarrow \infty} \sum_{m=0}^N \langle \psi_N, Q\partial_m \psi_m \rangle = \frac{1}{2} + \frac{2}{\pi^2} \\ \mathcal{C}_2 &= \lim_{N \rightarrow \infty} \sum_{n=0}^N \sum_{m=0}^N \langle \partial_n \psi_n, Q\partial_m \psi_m \rangle = \frac{1}{2} - \frac{1}{\pi^2}, \end{aligned} \quad (69)$$

which are worked out in [5], we can calculate the inner product

$$\begin{aligned} \langle \Psi, Q\Psi \rangle &= \lim_{N \rightarrow \infty} \left[\langle \psi_N, Q\psi_N \rangle - 2 \sum_{m=0}^N \langle \psi_N, Q\partial_m \psi_m \rangle + \sum_{n=0}^N \sum_{m=0}^N \langle \partial_n \psi_n, Q\partial_m \psi_m \rangle \right] \\ &= \mathcal{C}_0 - 2\mathcal{C}_1 + \mathcal{C}_2 \\ &= -\frac{3}{\pi^2}. \end{aligned} \quad (70)$$

The final result is

$$E = -\mathcal{T}_{25} = -\frac{1}{2\pi^2 g_o^2}, \quad (71)$$

and the first conjecture is proved.

4 Solution in the KBc Approach

There is another, simpler, but more abstract analytic solution to open string field theory that we shall discuss now. This approach uses what is often called the *KBc subalgebra*. The KBc subalgebra is at heart nicer packaging of the first proof. An equivalence between the two methods may be explicitly shown. Note that in the following discussion B and c are *not* the same as before.

4.1 K , B , and c

In this approach, we begin by defining

$$\begin{aligned} K_1 &\equiv L_1 + L_{-1} \\ B_1 &\equiv b_1 + b_{-1} , \end{aligned} \tag{72}$$

and defining an identity string field $|I\rangle$. K , B , and c are defined as

$$\begin{aligned} K &\equiv \frac{\pi}{2}(K_1)_L |I\rangle \\ B &\equiv \frac{\pi}{2}(B_1)_L |I\rangle \\ c &\equiv \frac{1}{\pi}c(1) |I\rangle , \end{aligned} \tag{73}$$

where the subscript L means integrating from $-i$ to i on the positive half of the unit circle. One can show the KBc subalgebra [13, 14, 15, 16] satisfies

$$\begin{aligned} [K, B] &= 0 \\ B^2 = c^2 &= 0 \\ \{B, c\} &= 1 . \end{aligned} \tag{74}$$

The product implicitly used here is the \star star product. The action of Q on K , B , and c is given by

$$\begin{aligned} QK &= 0 \\ QB &= K \\ Qc &= cKc . \end{aligned} \tag{75}$$

We can express the $SL(2, \mathbb{R})$ vacuum $|0\rangle$ as

$$\begin{aligned} |0\rangle &\equiv \Omega = e^{-K} \\ \Omega^t &= e^{-tK} . \end{aligned} \tag{76}$$

4.2 Ψ in the KBc approach

We suggest a solution Ψ to the equation of motion

$$Q\Psi + \Psi \star \Psi = 0. \quad (77)$$

The Ψ we suggest is

$$\Psi = [c + cKBc] \frac{1}{1+K}. \quad (78)$$

We use the *Schwinger parametrization* to define $\frac{1}{1+K}$ as

$$\frac{1}{1+K} = \int_0^\infty dt e^{-t(1+K)} = \int_0^\infty dt e^{-t\Omega}. \quad (79)$$

Note that in this form we see B commutes with $\frac{1}{1+K}$. It is now natural to express Ψ in a way that makes more mathematical sense,

$$\Psi = \int_0^\infty dt e^{-t} [c + cKBc] \Omega^t. \quad (80)$$

We now discuss two useful ways to rewrite the components of Ψ . Recall our solution candidate

$$\Psi = [c + cKBc] \frac{1}{1+K}. \quad (81)$$

Claim:

$$Q(Bc) = cKBc \quad (82)$$

Proof:

$$\begin{aligned} Q(Bc) &= QBc + (-1)^B BQc \\ &= Kc - BcKc \\ &= Kc - (1 - cB)Kc \\ &= cKBc \quad \blacksquare \end{aligned} \quad (83)$$

Claim:

$$c + cKBc = c(1+K)Bc \quad (84)$$

Proof:

$c = c(1-cB) = cBc$, since $c^2 = 0$ and $\{B, c\} = 1$. Rest follows naturally.

$$\begin{aligned} c + cKBc &= cBc + cKBc \\ &= (c + cK)Bc \\ &= c(1+K)Bc \quad \blacksquare \end{aligned} \quad (85)$$

4.3 Proof that Ψ satisfies the equations of motion

Now we have the tools to prove that the suggested Ψ satisfies the equation of motion. Begin by acting with Q on Ψ . Recall (82) and we obtain

$$\begin{aligned} Q\Psi &= Q[c + cKBc] \frac{1}{1+K} \\ &= [Qc + Q^2(Bc)] \frac{1}{1+K} \\ &= cKc \frac{1}{1+K}. \end{aligned} \tag{86}$$

Now compute $\Psi \star \Psi$. Begin by recalling the claim (84).

$$\begin{aligned} \Psi \star \Psi &= [c + cKBc] \frac{1}{1+K} (\star) [c + cKBc] \frac{1}{1+K} \\ &= c(1+K)Bc \frac{1}{1+K} c(1+K)Bc \frac{1}{1+K} \end{aligned} \tag{87}$$

Now, commute the B s toward one another and simplify.

$$\begin{aligned} \Psi \star \Psi &= c(1+K)(1-cB) \frac{1}{1+K} (1-Bc)(1+K)c \frac{1}{1+K} \\ &= c(1+K)(1-cB) \frac{1}{1+K} [(1+K) - Bc(1+K)] c \frac{1}{1+K} \\ &= c[(1+K) - (1+K)cB] \left[1 - \frac{1}{1+K} Bc(1+K)\right] c \frac{1}{1+K} \\ &= c[(1+K) - Bc(1+K) - (1+K)cB + (1+K)cB \frac{1}{1+K} Bc(1+K)] c \frac{1}{1+K} \end{aligned} \tag{88}$$

We can eliminate the final term in brackets by again commuting the B s toward one another and noting that $B^2 = 0$. Simplify again, noting $c^2 = 0$ and $c = cBc$, to obtain the final result.

$$\begin{aligned} \Psi \star \Psi &= c[(1+K) - (1-cB)(1+K) - (1+K)cB] c \frac{1}{1+K} \\ &= c[cB(1+K) - (1+K)cB] c \frac{1}{1+K} \\ &= [(-c - cK)cB] c \frac{1}{1+K} \\ &= -cKc \frac{1}{1+K} \end{aligned} \tag{89}$$

Therefore, we have our desired result

$$Q\Psi + \Psi \star \Psi = 0. \tag{90}$$

4.4 Proof of the First Conjecture

We start with our confirmed solution

$$\Psi = [c + cK Bc] \frac{1}{1 + K}. \quad (91)$$

As discussed when we used the Schnabl Gauge, the proof of the first conjecture reduces to proving the statement

$$\langle \Psi, Q\Psi \rangle = -\frac{3}{\pi^2}. \quad (92)$$

We begin to compute this inner product using the trace function $\text{Tr}(\cdot) = \langle I, \cdot \rangle$.

$$\begin{aligned} \langle \Psi, Q\Psi \rangle &= \left\langle [c + cK Bc] \frac{1}{1 + K}, cKc \frac{1}{1 + K} \right\rangle \\ &= \text{Tr} \left([c + cK Bc] \frac{1}{1 + K} cKc \frac{1}{1 + K} \right). \end{aligned} \quad (93)$$

It is now convenient to use (80) and (82) to write our inner product as

$$\langle \Psi, Q\Psi \rangle = \int_0^\infty dt_1 \int_0^\infty dt_2 e^{-t_1 - t_2} \left[\text{Tr} \left(c\Omega^{t_1} cKc\Omega^{t_2} \right) - \text{Tr} \left(Q [Bc\Omega^{t_1} cKc\Omega^{t_2}] \right) \right]. \quad (94)$$

The trace of a BRST exact state vanishes, so we may remove the second term in the integrand. Therefore, we have

$$\langle \Psi, Q\Psi \rangle = \int_0^\infty dt_1 \int_0^\infty dt_2 e^{-t_1 - t_2} \text{Tr} \left(c\Omega^{t_1} cKc\Omega^{t_2} \right). \quad (95)$$

We can evaluate a trace function by identifying it with a correlator [14].

$$\text{Tr} \left(c\Omega^{t_1} cKc\Omega^{t_2} \right) = -\left(\frac{t_1 + t_2}{\pi} \right)^2 \sin^2 \frac{\pi t_1}{t_1 + t_2} \quad (96)$$

Now our inner product can be written as an analytic integral.

$$\langle \Psi, Q\Psi \rangle = -\int_0^\infty dt_1 \int_0^\infty dt_2 e^{-t_1 - t_2} \left(\frac{t_1 + t_2}{\pi} \right)^2 \sin^2 \frac{\pi t_1}{t_1 + t_2} \quad (97)$$

Make the substitutions

$$u = t_1 + t_2, \quad u \in [0, \infty) \quad (98)$$

$$v = \frac{t_1}{t_1 + t_2}, \quad v \in [0, 1] \quad (99)$$

$$dt_1 dt_2 = u \, du \, dv. \quad (100)$$

We can now write (97) in the form

$$\begin{aligned}\langle \Psi, Q\Psi \rangle &= -\frac{1}{\pi^2} \left(\int_0^\infty du u^3 e^{-u} \right) \left(\int_0^1 dv \sin^2 \pi v \right) \\ &= -\frac{3}{\pi^2},\end{aligned}\tag{101}$$

and the conjecture is proved.

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